Algebra: Mathematics to a Purpose

We students often think of Algebra as a complicated set of rules we must learn to get a grade in a subject we need to get into college. But it is more than that. In its many centuries long history, algebra has helped people understand and control the physical world around them. Indeed, many improvements to Algebra were made in connection with the Industrial Revolution.

Algebra was not invented all at once; it was invented in many steps over many centuries. Each of these improvements was made to help people perform mathematical tasks as part of their work even though they did not have a deep understanding of mathematics. Perhaps most importantly to people our age, algebra is still evolving. Notations continue to change so people can use ever more powerful tools for the calculations needed by larger and more complex problems in science, engineering, and industry.

Algebra synthesizes contributions from the best mathematicians working at different times and places for over 1500 years. These contributions include many notations, procedures, and tools intended to help people who do not understand why Algebra works but still must perform mathematical tasks. For example, algebra has given people a way of expressing and using the scientific and engineering principles important to the machines, structures, and industrial processes that make our civilization possible.

Many of the best mathematicians have always tried to put algebra and other mathematics into the hands of ordinary people. Throughout history, the best mathematicians were able to solve difficult algebra problems using little more than the power of their own minds, e.g., their visual imagination, memory, attention to detail, and reasoning. But, it was often difficult for them to explain to those with less mathematical ability how they thought about the problem, how they arrived at their answer, and why their answer was correct.

Before they could explain Algebra to their students, mathematicians often had to invent new notations which could be used to explain in simple terms ideas that had previously been complex. My timeline includes milestones marking the invention of many important notations including variables, the equals sign, equations, coordinate planes, graphs, and function notation. Many of these improved notations were invented in the 17th and 18th Centuries as mathematicians tried to help students who needed to build new kinds of machines for the Industrial Revolution. Other notational improvements like the use of matricies and computer programming languages were introduced in the 19th and 20th Centuries to help people compute solutions to large and complex problems.

In the 19th Century, people were hired to be human computers; really, that is what they were called. In the 19th Century, there was an attempt to replace human computers with mechanical ones. It was not until the 20th Century that people were able to use electronic computers to do the computation required for large, complex algebra problems.

One of the most important ideas in the history of algebra was the insight that equations could be graphed on a coordinate plane. There were many centuries of invention in algebra before people knew enough to graph equations. The idea of graphing equations, due to the 17th Century mathematician Pierre de Fermat, gave scientists and engineers a powerful tool for understanding a wide range of physical phenomena (Merzbach 320). As evidenced by the publication of many textbooks on Algebra during the early days of the Industrial Revolution, algebra was being used as a language for communicating and applying newly discovered physical principles to a vast range of inventions (Merzbach 381,314). When solving problems that required use of these equations, people were content to believe the principles based on general understanding of the science. In turn, people solved these equations by making unembarrassed use of tools such as sectors, slide rules, published tables of values, and calculators (Merzbach 290-295). People did not require themselves to become expert in either the physical science or the mathematics they were using. It was enough that they trusted the skill and integrity of those who were providing them with the equations and tools they needed.

Even before the Industrial Revolution, however, people were doing complex calculations as part of designing and building things. The ability to calculate solutions quickly and accurately depended on many inventions in arithmetic and algebra during the centuries before the Industrial Revolution. The first algebra problems were word problems and there were no general procedures for solving them. Around 250 CE a Greek, Diophantus, showed the importance of defining one unknown quantity in terms of other unknowns (Merzbach 161-162). About 600 years later, a Persian mathematician, Al-Khwarizmi, described procedures for solving a problem by simplifying it until the answer was self-evident (Merzbach 206-207). In these early centuries of Diophantus and Al-Khwarizmi, algebra problems and their solutions were expressed in ordinary language. People solved problems by modeling their work on similar, previously solved problems (Merzbach 162). In Europe, people did their calculations using Roman numerals and the abacus well in to the 13th Century (Merzbach 227-229).

In the 13th Century, Europe began to replace the cumbersome Roman numerals and abacus with a new system that it learned from merchants in North Africa and the Middle East (Merzbach 227-229). This new system used the same Hindu-Arabic system we use today. With this new system, people used the digits 0 through 9 and procedures for calculating with pen and paper (Merzbach 228).

It was the Italian city of Pisa, which operated a large trading empire throughout the Mediterranean, that introduced Europe to this modern system of numbers and procedures. The Merchants of Pisa, influenced by Leonardo Pisano Fibonacci, adopted the new system so they could better work with trading partners around the Mediterranean. Fibonacci had learned to use these numerals from Arab teachers when as a child he traveled the Middle East with his father, who was a Pisan merchant. Inspired by Fibonacci’s example, the teaching of arithmetic became hugely popular throughout Italy, with perhaps a thousand or more hand-written arithmetic texts being produced over the following three centuries (Merzbach 229-230). By the 15th Century the transition to Hindu-Arabic numerals was complete (Merzbach 228).

With compact, efficient procedures for arithmetic, the better mathematicians could begin to understand and describe important recurring patterns of calculation. For example, Gerolamo Cardano, often considered to be the greatest mathematician of the Renaissance, collected all that was known about this task of finding the roots of equations, taking square roots of numbers, and the use of negative numbers. However, because of remaining limitations in his notation, Cardano could not explain $\sqrt{-1}$, a number which shows up in the solutions of many quadratic equations (Merzbach 255).

Some additional improvements were provided by Robert Recorde, a Welsh physician and mathematician who attended both Oxford and Cambridge. Recorde was the first person anywhere to use the modern = sign. Recorde was also popularized the modern symbols + and -. Around the same time, other mathematicians introduced improved notations for fractions, exponents, and the radical sign for taking roots of numbers. Simon Stevin, a Dutch mathematician, introduced the modern notation for decimal fractions, further simplifying the representation of numbers and the procedures for calculating quantities (Merzbach 262). Because of all the improvements in notation, Stevin was able to solve any quadratic equation using a single pattern for his solution, a pattern we now call the quadratic formula. Today we would write the quadratic formula as $x= \frac{-b \pm \sqrt{b^{2}-4ac} }{2a}$, but Stevin would write it differently. In Stevin’s time, variables had not yet been invented (Merzbach 262-286).

During the 16th Century machinery became more complex and more powerful, and careful measurement and calculation were required to make sure the machines actually worked. To fill this need, mathematicians invented a number of calculating devices. Craftsmen used these devices in a wide variety of projects. Some built ambitious civil engineering projects for ever growing towns and cities (Agriculture). Some worked on problems in ship navigation. Others sought to harness wind and water power to drive machinery for milling and weaving. For example, in 1563 Rev. William Lee invented a mechanical device for knitting stockings. As the 16th Century closed, water and wind power was being harnessed on a large scale and machines were incorporating more complex combinations wedges, levers, wheels, and pulleys (Agriculture).

In all of these designs careful measurement and calculation were required to ensure the parts of complex machines fit together exactly. In the course of their work, people discovered that their procedures for multiplication and division were complex, time consuming, and error prone. In response to this need, many mathematical instruments were developed for automating these calculations (Merzbach 282). The sector was an important tool for performing multiplication and division. A sector does multiplication and division using the principle of similar triangles (Williams). Another popular tool, Napier bones, is based on the multiplication tables. The person doing the multiplication uses the Napier bones to find the partial products for each pair of digits. The product is then calculated by adding the partial products together (Napier). Another of John Napier’s discoveries, logarithms, gave a way of multiplying numbers by adding and subtracting carefully measured lengths. Each length represents the magnitude of the exponent used to calculate the inscribed number from a common base. In 1642, when he was 18, Pascal invented the first mechanical calculator. Pascal’s machine could add, subtract, multiply, and divide (Pascal). In all these cases, the idea was the same: automating calculation so so ordinary people to perform mathematical tasks as part of their work (Pascal).

As the Industrial Revolution of the 18th Century gained steam, more improvements in algebra were needed. Of particular important was the use of functions to show how a dependent variable changed as a function of time. Other important improvements were the variable, the literal equation, the coordinate plane, and functions. François Viète introduced the use of letters (what we call variables) to represent unknown quantities in equations. He was also the first to write literal equations in which unknown fixed quantities, called parameters, were distinguished from other unknown varying quantities. The combination of literal equations and calculating devices made it possible for craftsmen to effectively solve design problems by applying physical principles represented as equations (Merzbach 273-274).

Fermat’s idea of graphing equations opened the way for engineers to use equations and graphs to visualize the behavior of physical systems. Fermat discovered that equations of two unknowns could be graphed on a coordinate plane. He observed that the shape of the graph, whether curved or straight, could be predicted by analyzing properties of the equation. In other words, Fermat discovered that equations of the type written by Viète could be graphed on a coordinate plane similar to the ones Descartes used. But, instead of drawing geometric figures the way Descartes did, Fermat drew graphs which represented the set of possible solutions to an equation in two variables (Merzbach 320-321).

The Industrial Revolution began in the early 18th Century with the construction, in England, of the first factories driven by stream power. With machines using new technologies operating with greater force and speed, engineers and craftsmen faced more difficult design problems using newly understood physical principles. To succeed with their designs, inventors, craftsmen, and engineers had to learn to use algebra as it had been improved by Viète, Descartes, and Fermat. Colin Maclaurin, Robert Simson, and other respected mathematicians of the era wrote widely used textbooks on Algebra. Consistent with its practical origins, thes 18th Century Algebra books emphasized the calculations needed to solve algebra problems, including even the rules for multiplying negative numbers. (Merzbach 380).

As the Industrial Revolution evolved into the Machine Age, mathematicians discovered there were limits to the types of problems that could be solved algebraically. Niels Henrik Abel proved that fifth and higher order polynomials had no algebraic solution. This result meant no exact solution to these high-order polynomials were possible and other solution methods had to be found. The new methods required a systematic search along the real number line for a polynomial’s roots. These “numerical” methods required a great deal of repetitive calculation. Significantly, these numerical methods are the cornerstone of the modern computer-based tools use to analyze functions (Merzbach 475).

Charles Babbage tried design and build mechanical calculating engines to eliminate the risk of human error when creating the numerical tables for the important functions used by engineers, architects, mathematicians, astronomers, bankers, actuaries, journeymen, insurance brokers, statisticians, and navigators. These tables, often the size of books, recorded the values of a function for a large set of domain values. These tables were calculated, copied, checked and typeset by hand. They were often full of errors. Babbage tried to invent a machine that would do all that work, eliminating the possibility of human error (Analysis).

 Babbage never succeed in building his mechanical computer; the manufacturing processes of the day could not fabricate the carefully designed parts he needed. Nevertheless, his work was important. One of Babbage’s collaborators, Ada Lovelace, understood better than any other person in her day, even better than Babbage himself, the true potential of the invention. Lovelace

speculated that the machine might go beyond numbers and more generally manipulate symbols in accordance with rules. She saw that numbers could represent entities other than quantity – letters of the alphabet, notes of music – and that by manipulating numbers, computing machines could extend their powers beyond the world of mathematics (Analysis).

Lovelace’s vision was 100 years ahead of its time.

In the 1940s, Babbage and Lovelace’s vision for a computational engine was realized using electronic instead of mechanical components. An electronic computer allowed program instructions to be treated as if they were data. This made is possible for the computer to be used as a tool in its own programming (Von Neumann). Developers saw it would be possible to use the computer to translate algebraic expressions into the low-level instructions that the computer would be able to execute. This translation from a high-level mathematical language into the instructions that an electronic device could understand was exactly the type of symbolic computing Ada Lovelace foresaw in the early 19th Century.

Advances in the practical solution of algebra problems continue to this day. In the 1950s, as soon as electronic computers became available, people built programming systems that could directly execute algebraic functions (Roots of Lisp). The last 20 years saw the invention of Computer Algebra systems capable of solving and analyzing equations symbolically. Computer Algebra systems can simplify and solve an equation using steps similar to those which might be taught to an algebra student. Computer Algebra systems can graph and analyze equations (Computer Algebra). The future is even more promising. Wolfram Alpha, which was released on May 18, 2009, is an online service that answers factual queries directly by computing the answer from externally sourced “curated data.” Its computational capabilities include computer algebra, symbolic and numerical computation, visualization, and statistics. It is noteworthy in that when constructing an answer to user’s question it can use mathematics to analyze data from diverse domains (WolframAlpha).

Computer algebra systems were originally designed for mathematicians, scientists, and engineers. Educators have little experience integrating their use into the classroom and little is known about how their use might affect a student learning in algebra. Research suggests that the use of computer algebra systems in the classroom can be effective if its use is carefully planned. For example, with the use of computer algebra systems, students may spend less time learning to solve one step, two step, and multi-step equations by hand. Instead, they might focus on discovery activities in which they explore deeper mathematical ideas connected with algebra or using the tools of algebra to investigate realistic, practical problems (Computer Algebra).

In the spirit of those who invented Algebra over the last 1500 years, educators and students must be bold. If we can step beyond old fashioned notions of the types of calculations which must be done by hand, we can free our imaginations to investigate interesting problems that have meaning in the practical world. We must not regret that computers can solve an equation for an unknown faster than we can. Instead, we must embrace freedom to use algebra to explore and understand the world around us.

Works Cited

"Agricultural and Industrial Revolution." Intriguing History. N.p., n.d. Web. 1 Apr. 2016. <http://www.intriguing-history.com/historic-themes/major-events-british-history/industrial-revolution/>.

"Blaise Pascal (1623~1662)." Blaise Pascal. N.p., n.d. Web. 03 Apr. 2016. <https://www. educalc.net/196488.page>.

"Computer Algebra Systems and Algebra: Curriculum, Assessment, Teaching, and Learning." ResearchGate. N.p., n.d. Web. 02 Apr. 2016. <http://www.researchgate.net/publication /226505592\_Computer\_Algebra\_Systems\_and\_Algebra\_Curriculum\_Assessment\_Teaching\_and\_Learning>.

"John Von Neumann: The Father of the Modern Computer." Devlin's Angle. N.p., n.d. Web. 02 Mar. 2016. <https://www.maa.org/external\_archive/devlin/devlin\_12\_03.html>.

Merzbach, Uta C., and Carl B. Boyer. A History of Mathematics. New York: Wiley, 2011. Print.

“Napier's Bones." -- from Wolfram MathWorld. N.p., n.d. Web. 01 Apr. 2016. <http://mathworld.wolfram.com/NapiersBones.html>

"The Introduction of Analysis into England." The Introduction of Analysis into England. N.p., n.d. Web. 28 Mar. 2016. <http://www.maths.tcd.ie/pub/HistMath/People /19thCentury/RouseBall/RB\_Engl19C. html#Babbage>. This source says it transcribed Babbage's biography from Ball, W. W. Rouse. A Short Account of the History of Mathematics. London: Macmillan, 1908. Print.

"The Roots of Lisp." The Roots of Lisp. N.p., n.d. Web. 02 Apr. 2016. <http://www. paulgraham.com/rootsoflisp.html>.

Williams, M.r., and E. Tomash. "The Sector: Its History, Scales, and Uses." IEEE Annals Hist. Comput. IEEE Annals of the History of Computing 25.1 (2003): 34-47. Web.

“Wolfram|Alpha: Computational Knowledge Engine.” Wolfram|Alpha: Computational Knowledge Engine. N.p., n.d. Web. 06 Apr. 2016. <https://www.wolframalpha.com /input/?i=wolfram%2Balpha>.